SELF-MODELING SOLUTIONS OF THE HEAT-CONDUCTION EQUATION

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Self-modeling solutions of the heat-conduction equation are described. The temperature moment is assumed constant.

Many self-modeling solutions are known for the nonlinear planar heat-conduction equation. In [1], solutions are constructed for boundary conditions in which the temperature or heat flow varies with some power of the time. In [2], a solution is obtained which is based on the assumption that the temperature moment $\int_0^\infty xT(x, t)dx$ is independent of time.

The present work generalizes the solutions found in [1] and [2]. It is found that many solutions, including those well-known can be obtained under the condition of constant temperature moment.

We multiply both sides of the nonlinear heat-conduction equation

$$\frac{\partial T^s}{\partial t} = \frac{\partial}{\partial x}\left(aT^n \frac{\partial T}{\partial x}\right) \tag{1}$$

by $t^p x^k T^m$ and integrate with respect to x from x = 0 (the medium boundary) to $x = x_1(t)$, where $x_1(t)$ is the thermal-wave front, at which the condition

$$T\,|_{x=x_1(t)} = aT^n \frac{\partial T}{\partial x}\bigg|_{x=x_1(t)} = 0. \tag{2}$$

is satisfied. We obtain

$$\frac{s}{m+s}\frac{d}{dt}\int_0^{x_1(t)} t^p x^k T^{m+s} dx = \frac{ps}{m+s}\int_0^{x_1(t)} t^{p-1} x^k T^{m+s} dx + \frac{a}{n+m+1} t^p \left(x^k \frac{\partial T^{n+m+1}}{\partial x} - kx^{k-1} T^{n+m+1}\right)\bigg|_0^{x_1(t)}$$
$$+ a\left[\frac{k(k-1)}{n+m+1}\int_0^{x_1(t)} t^p x^{k-2} T^{n+m+1} dx - m\int_0^{x_1(t)} t^p x^k T^{n+m-1}\left(\frac{\partial T}{\partial x}\right)^2 dx\right]. \tag{3}$$

All terms are assumed finite.

We seek those solutions of Eq. (1) for which the right side of Eq. (3) is zero. The condition

$$\int_0^{x_1(t)} t^p x^k T^{m+s} dx = M, \tag{4}$$

with M = const holds for these solutions. The parameters a from (1) and M from (4) are used to construct self-modeling solutions. They have the form

$$T(x, t) = (M^{\Delta_1} a^{\Delta_2} t^{\Delta_3})^{\frac{1}{\Delta}} f(\xi), \tag{5}$$

where

$$\xi = \frac{x}{(M^{\delta_1} a^{\delta_2} t^{\delta_3})^{\frac{1}{\Delta}}}, \tag{6}$$

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$$\begin{gathered}\Delta_1 = 2, \qquad \delta_1 = n - s + 1,\\ \Delta_2 = -(k+1), \quad \delta_2 = m + s,\\ \Delta_3 = -(k+1) - 2p, \quad \delta_3 = m + s - p(n - s + 1),\\ \Delta = 2(m+s) + (k+1)(n-s+1).\end{gathered} \tag{7}$$

Substituting (5) and (6) in Eq. (1), we obtain for $f(\xi)$ the ordinary differential equation

$$(f^n f')' + \frac{\delta_3 s}{\Delta} \xi f^{s-1} f' - \frac{\Delta_3 s}{\Delta} f^s = 0. \tag{8}$$

Certain auxiliary relationships are required in what follows. Integrating (1) with respect to x, and then (1) multiplied by x with respect to x, we obtain

$$\frac{d}{dt}\int_0^{x_1(t)} T^s dx = -aT^n \frac{\partial T}{\partial x}\Big|_{x=0}, \quad \frac{d}{dt}\int_0^{x_1(t)} T^s x dx = \frac{a}{n+1} T^{n+1}\big|_{x=0}.$$

Substituting T(x, t) from (5) and (6) in the latter equation, we have

$$-f^n f'\big|_{\xi=0} = \frac{\Delta_3 s + \delta_3}{\Delta}\int_0^{\xi_0} f^s(\xi)\, d\xi, \tag{9}$$

$$f^{n+1}\big|_{\xi=0} = (n+1)\,\frac{\Delta_3 s + 2\delta_3}{\Delta}\int_0^{\xi_0} f^s(\xi)\, \xi d\xi, \tag{10}$$

where ξ_0 is the value of the dimensionless parameter (6) at $x = x_1(t)$.

Equation (8) can be written in the form

$$(f^n f')' + \frac{\delta_3}{\Delta} \xi^{1+\omega} (\xi^{-\omega} f^s)' = 0, \tag{11}$$

where $\omega = \Delta_3 s/\delta_3$.

Integrating (11), we obtain

$$f^n f' + \frac{\delta_3}{\Delta} \xi f^s - \frac{\Delta_3 s + \delta_3}{\Delta}\int_0^{\xi_0} f^s(\xi)\, d\xi = C.$$

The constant C is determined by setting $\xi = 0$ and using (9):

$$f^n f' + \frac{\delta_3}{\Delta} \xi f^s + \frac{\Delta_3 s + \delta_3}{\Delta}\int_{\xi}^{\xi_0} f^s(\xi)\, d\xi = 0. \tag{12}$$

Including condition (2) at the front, we obtain the integral equation for

$$\frac{1}{n+1} f^{n+1}(\xi) = \frac{\delta_3}{\Delta}\int_{\xi}^{\xi_0} f^s(\xi)\, \xi d\xi + \frac{\Delta_3 s + \delta_3}{\Delta}\int_{\xi}^{\xi_0}\int_{\xi}^{\xi_0} f^s(\xi)\, d\xi d\xi. \tag{13}$$

The value of ξ_0 in (13) is unknown. The functions $f(\xi)$ and ξ_0 must satisfy the condition

$$\int_0^{\xi_0} f^{m+s}(\xi)\, \xi^k\, d\xi = 1, \tag{14}$$

in addition to (13). This follows from (4), using (5) and (6).

Qualitatively different solutions of(13) and (14) are obtained, depending on the relationships between the parameters sn,

$$s,\ n,\ m,\ k,\ p \tag{15}$$

Thus, for $\Delta_3 \geq 0$ we have solutions which satisfy the boundary condition $T(x, t)_{x=0} = At^{\alpha}$. Here the boundary heat flow is positive. It remains positive for $(\Delta_3 s + \delta_3)/\Delta > 0$. For $\Delta_3 s + \delta_3 = 0$ (zero heat flow at the boundary), the solution for $f(\xi)$ follows from (12):

$$f(\xi)=\left[\frac{n-s+1}{2}\frac{\delta_3}{\Delta}\xi_0^2\right]^{\frac{1}{n-s+1}}\left[1-\left(\frac{\xi}{\xi_0}\right)^2\right]^{\frac{1}{n-s+1}}, \tag{16}$$

where

$$\xi_0=\left[\frac{2}{B\left(\frac{k+1}{2},\ 1+\frac{m+s}{n-s+1}\right)}\right]^{\frac{n-s+1}{\Delta}}\left[\frac{n-s+1}{2}\frac{\delta_3}{\Delta}\right]^{-\frac{m+s}{\Delta}}. \tag{17}$$

For $\Delta_3 s + 2\delta_3 = 0$, the solution satisfies the boundary condition $T(x,t)_{x=0} = 0$. It is obtained from (11) without quadrature, in the form

$$f(\xi)=\left[(n+1)\frac{n-s+1}{n+s+1}\frac{\delta_3}{\Delta}\xi_0^2\right]^{\frac{1}{n-s+1}}\left(\frac{\xi}{\xi_0}\right)^{\frac{1}{n+1}}\left[1-\left(\frac{\xi}{\xi_0}\right)^{\frac{n+s+1}{n+1}}\right]^{\frac{1}{n-s+1}}, \tag{18}$$

where

$$\xi_0=\left(\frac{n+s+1}{n+1}\right)^{\frac{n+m+1}{\Delta}}\left[(n-s+1)\frac{\delta_3}{\Delta}\right]^{-\frac{m+s}{\Delta}}\left[B\left(1+\frac{m+k(n+1)}{n+s+1},\ 1+\frac{1}{n-s+1}\right)\right]^{-\frac{n-s+1}{\Delta}}. \tag{19}$$

In the expressions (17) and (19), $B(y, z)$ is the beta function.

The solutions (16) and (18) are generalizations of expressions derived in [1, 2]. They may also be obtained directly from (13). The integral equation (13) is also satisfied by the functions $f(\xi)$, corresponding to negative boundary heat flow at nonzero temperature. This occurs for

$$-\frac{\delta_3}{\Delta}<\frac{\Delta_3 s+\delta_3}{\Delta}<0.$$

Thus, a correspondence exists between the boundary conditions and the numerical values of the parameters (15). To obtain solutions of a given type, one must select the required parameter values.

Equation (13) can be solved numerically as follows. We divide the interval $[0, \xi_0]$ into equal segments by the points $\xi_0 > \xi_1 > \ldots > \xi_{N-1} > \xi_N = 0$. If $f^s(\xi)$ is replaced by a straight line in each segment, we obtain the following approximate equation after evaluating the integrals in (13):

$$f_q^{n+1}=f_{q-1}^{n+1}+(n+1)\frac{\delta_3}{\Delta}\frac{\xi_0^2}{2N}\left\{f_q^s\left[1-\frac{q}{N}+\left(2+\frac{\Delta_3 s}{\delta_3}\right)\frac{1}{3N}\right]\right.$$
$$\left.+f_{q-1}^s\left[1-\frac{q}{N}+\left(2+\frac{\Delta_3 s}{\delta_3}\right)\frac{2}{3N}\right]+\left(1+\frac{\Delta_3 s}{\delta_3}\right)\frac{1}{N}\sum_{i=0}^{q-2}(f_{i+1}^s+f_i^s)\right\}, \tag{20}$$

where $q = 1, 2, \ldots, N$; $f_q = f(\xi_q)$, and $f_0 = 0$.

A value must be assigned to ξ_0 for calculation purposes. The values of $f(\xi)$ and ξ_0 must satisfy Eq. (14). In conclusion, we note that Eqs. (5), (13), and (14) do not have meaning for all values of the parameters (15). Since the terms in (3) are bounded, it follows that if $T_{x=0} = 0$, then, using (18) we have

$$k>-\frac{m}{n+1}, \tag{21}$$

and if $T_{x=0} > 0$,

$$k>1. \tag{22}$$

Moreover, restrictions based on physical considerations are imposed on the parameters (15). The condition for wave-front velocity damping is obtained from (6):

$$1>\frac{\delta_3}{\Delta}>0. \tag{23}$$

The function $f(\xi)$ is positive for all values of the argument. We have, therefore, from (10)

$$\frac{\Delta_3 s+2\delta_3}{\Delta}\geqslant 0. \tag{24}$$

Since the thermal wave is propagated with finite velocity,

$$n>0. \tag{25}$$

It is also obvious that

$$s \geqslant 1. \tag{26}$$

NOTATION

T	is the temperature;
x	is the coordinate;
t	is the time;
a and M	are the dimensional parameters;
s, n, m, k, and p	are the dimensionless parameters.

LITERATURE CITED

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